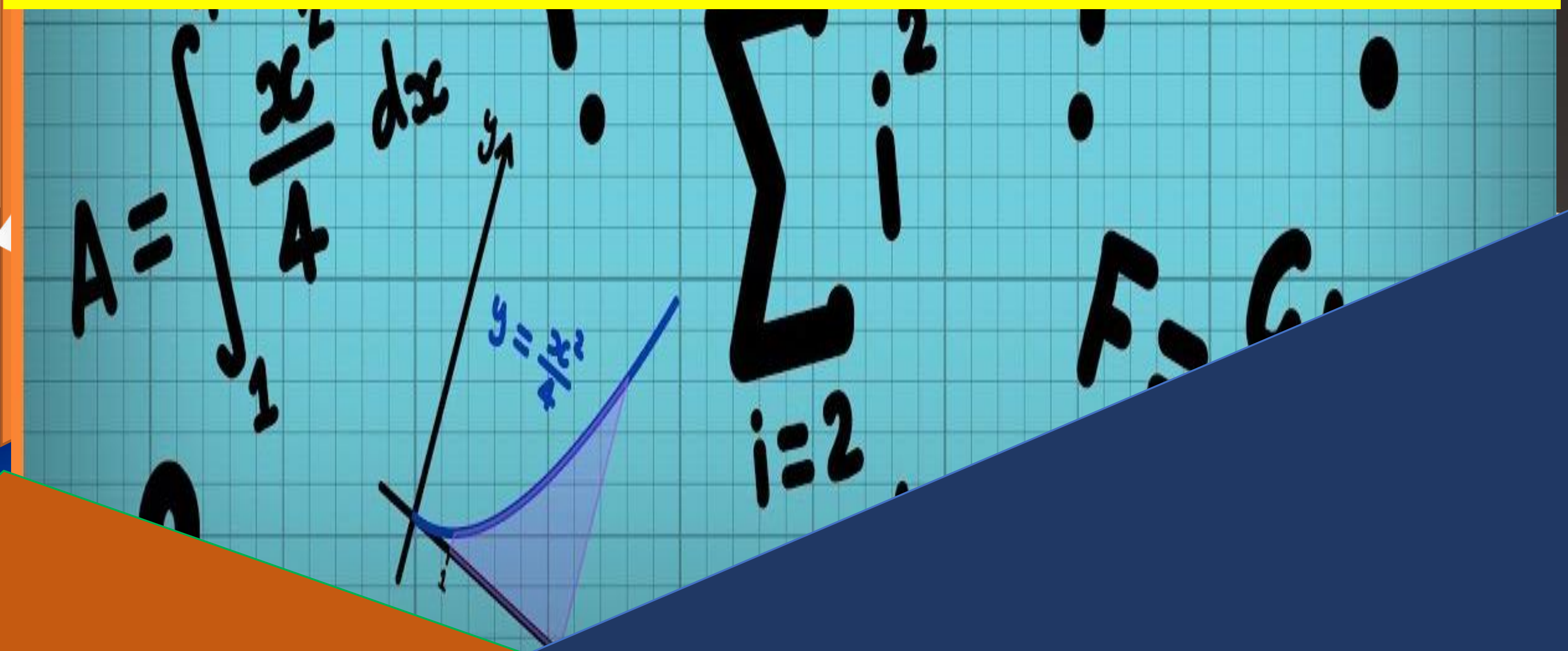
MATHEMATICS



CLASS-12

INVERSE TRIGONOMETRIC FUNCTIONS

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ASPIRING YOUNG MINDS

E SHIKSHA PIE PROGRAMS

CLASSES VI, VII, VIII, IX & X

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INVERSE TRIGONOMETRIC FUNCTIONS

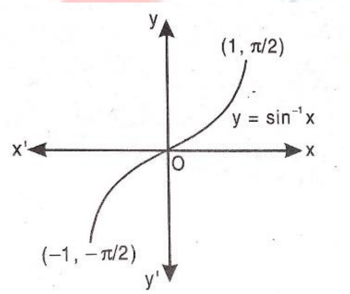
If $\sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$

But if $\sin\theta = 1/5$, in this condition θ is not well defined so, $\theta = \sin^{-1}(1/5)$.

Similarly, $\cos\theta = 1/7 \Rightarrow \theta = \cos^{-1}(1/7)$ and if,

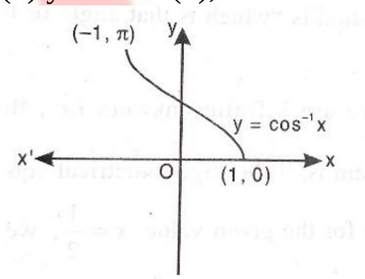
$\tan\theta = 13 \Rightarrow \theta = \tan^{-1}(13)$, $\tan\theta = m \Rightarrow \theta = \tan^{-1}(m)$

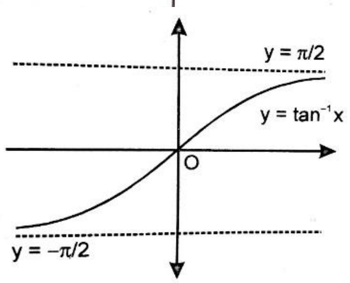
Value of I T F's



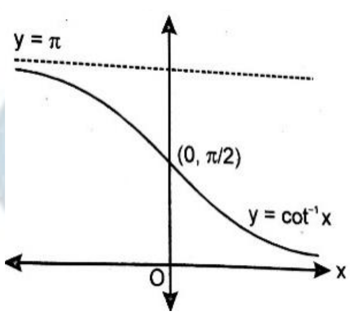
(a) $y = \sin^{-1}(\theta)$, Domain : $\theta \in [-1,1]$ and Range : $y \in [-\pi/2, \pi/2]$

(b) $y = \cos^{-1}(\theta)$, Domain : $\theta \in [-1, 1]$ and Range : $y \in [0, \pi]$.



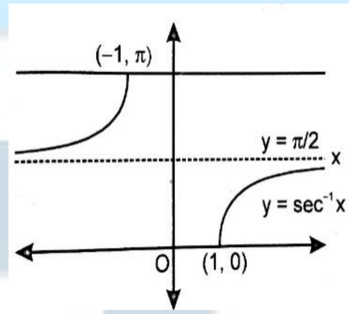


(c) $y = \tan^{-1}(\theta)$, Domain : $\theta \in R$ and Range : $y \in (-\pi/2, \pi/2)$



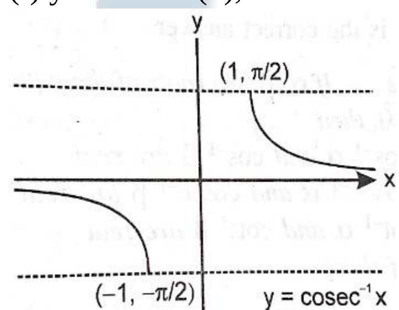
(d) $y = \cot^{-1}(\theta)$, Domain : $\theta \in R$ and Range : $y \in (0, \pi)$

(e) $y = \sec^{-1}(\theta)$,



Domain : $\theta \in (-\infty, -1] \cup [1, \infty)$ and Range: $y \in [0, \pi/2) \cup (\pi/2, \pi]$

(f) $y = \text{cosec}^{-1}(\theta)$, Domain : $\theta \in (-\infty, 1] \cup [1, \infty)$ and Range : $y \in [-\pi/2, 0) \cup (0, \pi/2]$



S. No.	Function	Domain	Range
1.	$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
2.	$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
3.	$\tan^{-1}x$	R	$(-\pi/2, \pi/2)$
4.	$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$
5.	$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$
6.	$\cot^{-1}x$	R	$(0, \pi)$

S. No.	Function	Principal Value branch
1.	$\sin^{-1}x$	$-\pi/2 \le y \le \pi/2$, where $y = \sin^{-1}x$
2.	$\cos^{-1}x$	$0 \le y \le \pi$, where $y = \cos^{-1}x$
3.	$\tan^{-1}x$	$-\pi/2 < y < \pi/2$, where $y = \tan^{-1}x$
4.	$\text{cosec}^{-1}x$	$-\pi/2 \le y \le \pi/2$, where $y = \text{cosec}^{-1}x$, $y \ne 0$
5.	$\sec^{-1}x$	$0 \le y \le \pi$, where $y = \sec^{-1}x$, $y \ne \pi/2$
6.	$\cot^{-1}x$	$0 < y < \pi$, where $y = \cot^{-1}x$

Relation between I T F's

(a) $\sin^{-1} x = \text{cosec}^{-1}(1/x)$,

(b) $\cos^{-1} x = \sec^{-1}(1/x)$,

(c) $\tan^{-1} x = \cot^{-1}(1/x)$

(d) $\sin^{-1} x + \cos^{-1} x = \pi/2$,

(e) $\tan^{-1} x + \cot^{-1} x = \pi/2$,

(f) $\sec^{-1} x + \text{cosec}^{-1} x = \pi/2$

SOME RESULTS

(a) $\sin^{-1}(\sin\theta) = \theta = \sin(\sin^{-1}\theta)$,

$\cos^{-1}(\cos\theta) = \theta = \cos(\cos^{-1}\theta)$

(b) (i) $\sin^{-1}(-x) = -\sin^{-1}(x)$,

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

(iii) $\tan^{-1}(-x) = -\tan^{-1}(x)$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$

Some useful formulas for I T F's

(a) If $x > 0, y > 0$ then;

$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\{(x \pm y)/(1 \mp xy)\}$, if $xy < 1$.

(b) If $x > 0, y > 0$ then:

$\tan^{-1} x + \tan^{-1} y$

$= \pi + \tan^{-1}\{(x + y)/(1 + xy)\}$, if $xy > 1$.

(c) If $x < 0, y < 0$ then:

$\tan^{-1} x + \tan^{-1} y$

$= \tan^{-1}\{(x + y)/(1 - xy)\} - \pi$, if $xy > 1$.

(d) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$= \tan^{-1}\{(x + y + z - xyz)/(1 - xy - yz - zx)\}$

(e) $\cot^{-1} x + \cot^{-1} y = \cot^{-1}\{(xy - 1)/(x + y)\}$

(f) $\cot^{-1} x - \cot^{-1} y = \cot^{-1}\{(xy + 1)/(y - x)\}$

(g) $\sin^{-1} x + \sin^{-1} y$

$= \sin^{-1}\{x\sqrt{(1 - y^2)} + y\sqrt{(1 - x^2)}\}\ \forall\ x \geq 0$

and $y \geq 0,\ x^2 + y^2 \leq 0$

(h) $\sin^{-1} x + \sin^{-1} y$

$= \pi + \sin^{-1}\{x\sqrt{(1 - y^2)} + y\sqrt{(1 - x^2)}\}\forall\ x \geq 0$

and $y \geq 0,\ x^2 + y^2 \leq 1$

(i) $\sin^{-1}x - \sin^{-1}y$

$= \sin^{-1}\{x\sqrt{(1 - y^2)} - y\sqrt{(1 - x^2)}\}\forall x \geq y \geq 0$

(i) $\cos^{-1} x + \cos^{-1} y$

$= \cos^{-1}\{xy - \sqrt{(1 - x^2)}\sqrt{(1 - y^2)}\}\forall x \geq 0, y \geq 0$

(j) $\cos^{-1}x - \cos^{-1}y$

$= \cos^{-1}\{xy + y\sqrt{(1 - x^2)}\sqrt{(1 - y^2)}\}\forall x \geq y \geq 0$

If x > 0, then

(a) $\sin^{-1} x = \cos^{-1}\sqrt{(1 - x^2)} = \tan^{-1}\{x/\sqrt{(1 - x^2)}\}$

(b) $\cos^{-1}x = \sin^{-1}\sqrt{(1 - x^2)} = \tan^{-1}\sqrt{(1 - x^2)}/x$

(c) $\tan^{-1}x = \sin^{-1}\{x/\sqrt{(1+x^2)}\} = \cos^{-1}\{1/\sqrt{(1+x^2)}\}$

(d) $2\tan^{-1}x = \tan^{-1}\{2x/(1 - x^2)\}\ \forall\ x \in (-1,1)$

(e) $2\tan^{-1}x = \cos^{-1}\{(1-x^2)/(1+x^2)\}\ \forall x \in [-1,1]$

(f) $2\tan^{-1}x = \sin^{-1}\{2x/(1 + x^2)\}\ \forall x \in [-1,1]$

(g) $2\sin^{-1}x = \sin^{-1}\{2x\sqrt{(1 - x^2)}\}$

(h) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

(i) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

(j) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(k) $3\tan^{-1}x = \tan^{-1}\{(3x - x^3)/(1 - 3x^2)\}$

(*l*) $3\cot^{-1}x = \cot^{-1}\{x^3 - 3x\}/(3x^2 - 1)$

Note: (Prove all the above formulae).

deg	0^0	30°	45°	60°	90°
Rad	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	½	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$\propto$

NOTE: Substitution forms

If we have	Put
1. $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
2. $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\text{cosec}\theta$
3. $\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
4. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$	$x = a\cos\theta$
5. $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos\theta$
6. $\sqrt{a/(x+a)}$ or $\sqrt{(x+a)/a}$	$x = a\tan^2\theta$

7. $\frac{1+tanx}{1-tanx} = \frac{cosx+sinx}{cosx-sinx} = \tan(\pi/4 + x)$

8. $1 + \cos x = 2\cos^2 x/2$

9. $1 - \cos x = 2\sin^2 x/2$

10. $\sin x = 2\sin(x/2)\cos(x/2)$, $1 = \sin^2 x + \cos^2 x$.

EXAMPLE – 1

Write the principal values of the following:

(a) $\sin^{-1}(1/2)$ (b) $\tan^{-1}1$ (c) $\sec^{-1}(2/\sqrt{3})$

Solution:

(a) $\sin\pi/6 = ½ \Rightarrow \sin^{-1} ½ = \pi/6$. Ans.

(b) $\tan\pi/4 = 1 \Rightarrow \tan^{-1} 1 = \pi/4$. Ans.

(c) $\sec\pi/6 = \frac{2}{\sqrt{3}} \Rightarrow \sec^{-1} 2/\sqrt{3} = \pi/6$. Ans.

EXAMPLE – 2

Write the principal values of the following:

(a) $\cos^{-1}(-1/2)$ (b) $\tan^{-1}(-1)$ (c) $\sec^{-1}(-2/\sqrt{3})$

Solution:

(a) $\cos 2\pi/3 = -½$

$\Rightarrow \cos^{-1}(-½) = \pi - \pi/3 = 2\pi/3$. Ans. ($2^{nd}$ –Q)

(b) $\tan(-\pi/4) = -1 \Rightarrow \tan^{-1}(-1) = -\pi/4$. Ans ($4^{th}$–Q)

(c) $\sec 5\pi/6 = -\frac{2}{\sqrt{3}}$

$\Rightarrow \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \pi - \pi/6 = 5\pi/6$. Ans. ($2^{nd}$ – Q)

EXAMPLE – 3

Find the value of the following:

(a) $2\cos^{-1} ½ + 3\sin^{-1}(1/2)$

(b) $\tan^{-1}1 + \cos^{-1}(-1/2) + \sin^{-1}(-1/2)$

Solution:

(a) $2.\pi/3 + 3.\pi/6 = 7\pi/6$. Ans.

(b) $\pi/4 + \pi - \pi/3 - \pi/6 = 3\pi/4$. Ans.

EXAMPLE – 4

Find the value of the following:

(a) $\sin^{-1}(\sin 2\pi/3)$ (b) $\cos^{-1}(\cos 13\pi/6)$

(c) $\cot^{-1}(\cot(-\pi/4))$ (d) $\tan^{-1}(\sin(-\pi/2))$

(e) $\cos^{-1}(\cos 2\pi/3) + \sin^{-1}(\sin 2\pi/3)$

Solution:

(a) $\sin^{-1}(\sqrt{3}/2) = \pi/3$ ($2\pi/3$, 2^{nd} Q)

(b) $\cos^{-1}(\sqrt{3}/2) = \pi/6$. ($13\pi/6$, 1^{st} Q)

(c) $\cot^{-1}(-1) = \pi - \pi/4 = 3\pi/4$ (2^{nd} Q)

(d) $\tan^{-1}(-1) = -\pi/4$. (4^{th} Q)

(e) $2\pi/3 + \pi/3 = \pi$. (2^{nd} Q)

EXAMPLE – 5

Evaluate:

(a) $\cos^{-1}(\cos 2\pi/3) + \sin^{-1}(\sin 2\pi/3)$

(b) $\tan^{-1}(2\cos(2\sin^{-1} 1/2))$.

Solution:

(a) $\cos^{-1}(\cos 4\pi/3) + \sin^{-1}(\sin 5\pi/3) = 2\pi/3 - \pi/3 = \pi/3$.

(b) $\tan^{-1}(2\cos(2\ .\ \pi/6)) = \tan^{-1}(2\cos\pi/3)$

$= \tan^{-1}(2.1/2) = \tan^{-1}(1) = \pi/4$.

Try yourself:

Evaluate:

(a) $\sin^{-1}(-1)$ Ans: $-\pi/2$.

(b) $\cot^{-1}(-\sqrt{3})$ Ans: $5\pi/6$

(c) $\sin^{-1}(1/2) + \cos^{-1}(\frac{\sqrt{3}}{2}) + \tan^{-1}(1)$ Ans: ($7\pi/12$)

EXAMPLE – 6

Prove the following:

(a) $\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

(b) $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

(c) If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2}$, prove that: $xy = 1$.

Solution:

(a) $\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Let $\sin^{-1}\frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

Now, $\cos x = \sqrt{1-\sin^2 x} = \frac{4}{5}$

So, $\tan x = \left(\frac{\sin x}{\cos x}\right) = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$

Hence $\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$ Ans.

(b) $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

LHS $= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right) = \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1} = \frac{\pi}{4}$ = RHS

(c) If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2}$, prove that $xy = 1$.

$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{2} \Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{2} = \infty$

$\Rightarrow \frac{x+y}{1-xy} = \frac{1}{0} \Rightarrow xy = 1$. Proved.

EXAMPLE – 7

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, then prove that:

$xy + yz + zx = 1$.

Solution:

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2} \Rightarrow \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan\frac{\pi}{2} = \frac{1}{0} \Rightarrow xy + yz + zx = 1$ Proved.

EXAMPLE – 8

If $\tan^{-1}x + \tan^{-1}(x+1) + \tan^{-1}(x-1)$

$= \tan^{-1}(3x)$, then find the value of x.

Solution:

$\tan^{-1}x + \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(3x)$

$\Rightarrow \tan^{-1}\left[\frac{x+x+1+x-1-x(x+1)(x-1)}{1-x(x+1)-(x+1)(x-1)-x(x-1)}\right] = \tan^{-1} 3x$

$\Rightarrow \frac{4x-x^2}{2-3x^2} = \frac{3x}{1} \Rightarrow 2x(4x^2-1) = 0 \Rightarrow x = 0, \pm\frac{1}{2}$.

EXAMPLE – 9

If $\tan^{-1}(x)^{-1} = \cot^{-1}\left(\frac{4}{x}\right)$, hence find x.

Solution:

$\tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{x}{4}\right) \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ Ans.

EXAMPLE – 10

If $\sin^{-1}x + \sin^{-1}y = 2\pi/3$, then find the value of

$\cos^{-1}x + \cos^{-1}y$.

Solution:

$\sin^{-1}x + \sin^{-1}y = 2\pi/3 \Rightarrow \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y = 2\pi/3$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - 2\pi/3 = \pi/3.$

Try yourself:

(a) Prove that: $\sin^{-1}x + \cos^{-1}x = \pi/2$

(b) $\tan^{-1}\frac{\cos x}{1+\sin x}$. Ans: $\pi/4 - x/2$.

(c) Prove that: $\tan^{-1}(2/11) + \tan^{1}(7/24) = \tan^{-1}(1/2)$.

(d) $\sin(\pi/3 - \sin^{-1}(-1/2))$ Ans: 1

(e) Prove that if, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then prove that: $x + y + z = xyz$.

EXAMPLE – 11

Prove that: $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = \pi/2$.

Solution:

Using the formula: $\sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$. We get:

$$\text{LHS} = \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2} + \frac{2}{\sqrt{5}}\sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2}\right]$$

$$\Rightarrow \sin^{-1}\left[\frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} \times \frac{2}{\sqrt{5}}\right]$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{5} + \frac{4}{5}\right) = \sin^{-1}(1) = \frac{\pi}{2} = \text{RHS} .$$

EXAMPLE – 12

Prove that: $2\tan(\tan^{-1}x + \tan^{-1}x^3) = \tan(2\tan^{-1}x)$

Solution:

$$\text{LHS: } 2\tan\left[\tan^{-1}\left(\frac{x+x^3}{1-x^4}\right)\right]$$

$$= 2\tan\left[\tan^{-1}\left(\frac{x(x^2+1)}{(1+x^2)(1-x^2)}\right)\right] = 2\left(\frac{x}{1-x^2}\right) = \frac{2x}{1-x^2}$$

RHS: $\tan(2\tan^{-1}x)$

$$= \tan\left[\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] = \frac{2x}{1-x^2}.$$ Proved.

EXAMPLE – 13

Solve: $4\sin^{-1}x + \cos^{-1}x = \pi$.

Solution:

$4\sin^{-1}x + \frac{\pi}{2} - \sin^{-1}x = \pi \Rightarrow 3\sin^{-1}x = \pi/2$

$\Rightarrow \sin^{-1}x = \pi/6 \Rightarrow x = 1/2$ Ans.

EXAMPLE – 14

Solve for x, y: $\sin^{-1}x + \sin^{-1}y = 2\pi/3$ and $\cos^{-1}x - \cos^{-1}y = \pi/3$.

Solution:

Let $\sin^{-1}x + \sin^{-1}y = 2\pi/3$……………..(1)

and $\cos^{-1}x - \cos^{-1}y = \pi/3$……………...(2)

By eq (1),

$\frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y = \frac{2\pi}{3}$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$…………………(3)

$\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$

Adding: $2\cos^{-1}x = \frac{2\pi}{3} \Rightarrow \cos^{-1}x = \frac{\pi}{3}$

$\Rightarrow x = \frac{1}{2}$. On solving eq (1), we get:

$y = 1$. So, $x = \frac{1}{2}$, $y = 1$ Answer.

EXAMPLE – 15

Prove that:

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{cb+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0.$$

Solution:

We have: $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{cb+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$

$\Rightarrow \cot^{-1}b - \cot^{-1}a + \cot^{-1}c - \cot^{-1}b + \cot^{-1}a - \cot^{-1}c = 0$

$= \text{RHS}.$

EXAMPLE – 16

If $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2p}{1+p^2}\right) + \cos^{-1}\left(\frac{1-q^2}{1+q^2}\right)$, then prove that: $x = \frac{p+q}{1-pq}$.

Solution:

$2\tan^{-1}x = 2\tan^{-1}p + 2\tan^{-1}q$

$\Rightarrow \tan^{-1}x = \tan^{-1}\left(\frac{p+q}{1-pq}\right) \Rightarrow x = \frac{p+q}{1-pq}$. Proved.

EXAMPLE – 17

Prove that: $\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$.

Solution:

Let $x^2 = \cos2\theta$. So, LHS:

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right]$$

$$\Rightarrow \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2 = \text{RHS}$$

EXAMPLE – 18

Show that: $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}] = \sqrt{\frac{x^2+1}{x^2+2}}$.

Solution:

Let $x = \tan\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}}$.

LHS $= \sin\cot^{-1}\cos\theta = \sin\cot^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$.

Let: $\frac{1}{\sqrt{1+x^2}} = \cot\alpha$

$\Rightarrow \sin\alpha = \frac{\sqrt{x^2+1}}{\sqrt{1+x^2+1}} \Rightarrow \sin\alpha = \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$ = RHS.

Try yourself

1. Find x: $2\tan^{-1}x = \tan^{-1}(1/4)$. Ans: $4 \pm \sqrt{17}$.
2. Evaluate: $\cos^{-1}(\cos(13\pi/6))$. Ans: $\pi/6$.
3. Prove that: $2\sin^{-1}(3/5) = \tan^{-1}(24/7)$.

EXAMPLE – 19

Show that: $\sum_{r=1}^{\infty} \tan^{-1}\left[\frac{1}{1+r+r^2}\right] = \frac{\pi}{4}$.

Solution:

$$\text{LHS} = \sum_{r=1}^{\infty} \tan^{-1}\left[\frac{1}{1+r+r^2}\right]$$

$$= \sum_{r=1}^{\infty}[\tan^{-1}(r+1) - \tan^{-1}r]$$

$= (\tan^{-1}2 - \tan^{-1} 1) + (\tan^{-1}3 - \tan^{-1} 2)$
$+ (\tan^{-1}4 - \tan^{-1} 3) + \ldots\ldots$

$= - \tan^{-1}1 + \tan^{-1}(\infty) = \pi/4 = \text{RHS}$.

EXAMPLE – 20

Find the sum of the series:

$\tan^{-1}\left(\frac{1}{1+1+1^2}\right) + \tan^{-1}\left(\frac{1}{1+2+2^2}\right) + \tan^{-1}\left(\frac{1}{1+3+3^2}\right) + \ldots\ldots$

Solution:

$T_r = \tan^{-1}\left[\frac{1}{1+r+r^2}\right] = \tan^{-1}\left[\frac{(r+1)-r}{1+r(r+1)}\right] = \tan^{-1}(r+1) - \tan^{-1}r$

$S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n = \tan^{-1}(n+1) - \tan^{-1}(1)$

$= \tan^{-1}\left[\frac{(n+1)-1}{1+1.(n+1)}\right] = \tan^{-1}\left(\frac{n}{n+2}\right)$ Ans.

EXAMPLE – 21

Solve the equation: $6\sin^{-1}(x^2 - 6x + 17/2) = \pi$.

Solution:

We have: $6\sin^{-1}(x^2 - 6x + 17/2) = \pi$

$\Rightarrow \sin^{-1}(x^2 - 6x + 17/2) = \frac{\pi}{6}$

$\Rightarrow \left(\frac{2x^2 - 12x + 17}{2}\right) = \frac{1}{2} \Rightarrow 2x^2 - 12x + 16 = 0$

$\Rightarrow 2(x^2 - 6x + 8) = 0 \Rightarrow (x-4)(x-2) = 0$

$\Rightarrow x = 2, 4$ Ans.

EXAMPLE – 22

Solve for x: $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec}x)$.

Solution:

LHS: $2\tan^{-1}(\cos x) = \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)$

$\Rightarrow 2\tan^{-1}(\cos x) = \tan^{-1}(2\cot \operatorname{cosec} x)$

So, $\tan^{-1}(2\cot x \operatorname{cosec}x) = \tan^{-1}(2\operatorname{cosec}x)$

$\Rightarrow \cot x = 1 \Rightarrow x = \pi/4$ Ans

And G. Solution is: $x = n\pi + \pi/4$ Ans.

EXAMPLE – 23

If $(\cot^{-1} x)^2 - 5(\cot^{-1} x) + 6 > 0$, then find x.

Solution:

Factorization form is: $(\cot^{-1}x - 2)(\cot^{-1}x - 3) > 0$

$\Rightarrow \cot^{-1}x < 2$ & $\cot^{-1}x > 3$

$\Rightarrow x \in (-\infty, \cot 3) \cup (\cot 2, \infty)$. Ans.

EXAMPLE – 24

Evaluate: $\cos^{-1}x + \cos^{-1}\left[\frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right], x \in \left[\frac{1}{2}, 1\right]$.

Solution:

$\cos^{-1}x + \cos\left[\frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right]$, putting $x = \cos\theta$, so

$\cos^{-1}x + \cos^{-1}\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)$

$= \cos^{-1}x + \cos^{-1}\cos\left(\frac{\pi}{3} - \theta\right) = \cos^{-1}x + \frac{\pi}{3} - \theta$

$= \cos^{-1}x + \frac{\pi}{3} - \cos^{-1}x = \frac{\pi}{3}$. Ans.

EXAMPLE – 25

Prove that:

$\tan^{-1}x = 2\tan^{-1}[\operatorname{cosec}\cot^{-1}x^{-1} - \tan\cot^{-1}x]$.

Solution:

RHS $= 2\tan^{-1}[\operatorname{cosec}\cot^{-1}x^{-1} - \tan\cot^{-1}x]$

$= 2\tan^{-1}[\operatorname{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x)]$

$= 2\tan^{-1}\left[\operatorname{cosec}\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x} - \tan\tan^{-1}\frac{1}{x}\right]$

$= 2\tan^{-1}\left(\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right)$.

Putting $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$= 2\tan^{-1}\left[\frac{\sec\theta}{\tan\theta} - \frac{1}{\tan\theta}\right] = 2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right] = 2\tan^{-1}\left[\tan\frac{\theta}{2}\right]$

$\Rightarrow \theta = \tan^{-1}x = \text{LHS}$.

EXAMPLE – 26

If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = 3\pi/2$, then prove that:

$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101}+y^{101}+z^{101}} = 0$.

Solution:

We have: $\sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \pi/2$.

So, $x = y = z = 1$. Now:

$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101}+y^{101}+z^{101}}$

$= 1 + 1 + 1 - 9/(1 + 1 + 1) = 3 - 3 = 0$.

EXAMPLE – 27

If $\tan^{-1}(a/x) + \tan^{-1}(b/x) = \pi/2$, find the value of x.

Solution:

We have: $\tan^{-1}(a/x) + \tan^{-1}(b/x) = \pi/2$

$\tan^{-1}\left(\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{xx}}\right) = \frac{\pi}{2} \Rightarrow \frac{(a+b)x}{x^2-ab} = \tan(\pi/2) = \infty$

So, $x^2 - ab = 0 \Rightarrow x = \sqrt{ab}$.

EXAMPLE – 28

Prove that: $\tan^{-1}\frac{6x-8x^3}{1-12x^2} - \tan^{-1}\frac{4x}{1-4x^2} = \tan^{-1}2x$.

Solution:

We have LHS: $\tan^{-1}\frac{6x-8x^3}{1-12x^2} - \tan^{-1}\frac{4x}{1-4x^2}$

$= 3\tan^{-1}2x - 2\tan^{-1}2x = \tan^{-1}2x$ = RHS.

EXAMPLE – 29

If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = 5\pi^2/8$, then find x.

Solution:

We have:

$(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = 5\pi^2/8$

$\Longrightarrow (\tan^{-1}x)^2 + (\pi/2 - \tan^{-1}x)^2 = 5\pi^2/8$

$\Longrightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x = 5\pi^2/8 - \pi^2/4 = 3\pi^2/8$

$\Longrightarrow 16(\tan^{-1}x)^2 - 8\pi\tan^{-1}x - 3\pi^2 = 0$

By formula method: $\tan^{-1}x = \frac{8\pi \pm 16\pi}{32} = \frac{3\pi}{4}, -\frac{\pi}{4}$.

So, x = - 1. Ans.

EXAMPLE – 30

Let $a\tan^{-1}\left(\frac{2x}{1-x^2}\right) + b\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + c\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi/2$, where a + b + c = 1, then find x:

Solution:

We have:

$a\tan^{-1}\left(\frac{2x}{1-x^2}\right) + b\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + c\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi/2$

$\Longrightarrow a(2\tan^{-1}x) + b(2\tan^{-1}x) + c(2\tan^{-1}x) = \pi/2$

$\Longrightarrow 2\tan^{-1}x\,(a + b + c) = \pi/2$ (Put a + b + c = 1)

$\Longrightarrow \tan^{-1}x = \pi/4 \Longrightarrow x = \tan(\pi/4) = 1$. Ans.

PROBLEMS

1. Find the principle of the following:

(a) $\cos^{-1}(\sqrt{3}/2)$ (b) $\tan^{-1}(-1)$

Ans: (a) $\pi/6$ (b) $-\pi/4$

(c) $\text{cosec}^{-1}(2)$ Ans: (c) $\pi/6$

(d) $\cot^{-1}(\sqrt{3})$ (e) $\tan^{-1}(-\sqrt{3})$

Ans: (d) $\pi/6$ (e) $-\pi/3$

(f) $\cos^{-1}(\cos\frac{13\pi}{6})$ (g) $\tan^{-1}(\tan\frac{9\pi}{8})$

Ans: (f) $\pi/6$ (g) $\pi/8$

2. Find the value of the following:

(a) $\tan^{-1}(3/4) = x$, find sinx and secx. Ans: 3/5, 5/4.

(b) $\sec(\cos^{-1}(1/2)) = k$, find k. Ans: 2.

(c) $\sin(\cot^{-1}x) = \lambda$, find λ. Ans: $\frac{1}{\sqrt{1+x^2}}$.

(d) $\tan^{-1}(x) = \theta$, find $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Ans: 2θ.

(e) $\sin[2\cot^{-1}(-5/12)]$ Ans: 120/169

3. Prove the following:

(a) $\sin^{-1}x + \cos^{-1}x = \pi/2$

(b) $\tan^{-1}x + \cot^{-1}x = \pi/2$

(c) $\text{cosec}^{-1}x + \sec^{-1}x = \pi/2$

4. Prove the following:

(a) $\frac{1}{2}\tan^{-1}\frac{4}{3} = \tan^{-1}\frac{1}{2}$

(b) $\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

(c) $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$

(d) $\sec^{-1}(\tan^{-1}1/3) + \text{cosec}^{-1}(\cot^{-1}3) = \pi/2$

(e) $2\tan^{-1}(1/3) + \tan^{-1}(1/7) = \pi/4$

(f) $\tan^{-1}a + \cot^{-1}(a + 1) = \tan^{-1}(a^2 + a + 1)$.

(g) $\tan^{-1}(n/(n+1)) - \tan^{-1}(2n+1) = 3\pi/4$

5. Prove that:

(a) $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

(b) $\tan^{-1}(5) - \tan^{-1}(3) + \tan^{-1}(7/9) = \pi/4$.

(c) $\cot(\text{cosec}^{-1}(5/3) + \tan^{-1}(2/3)) = 6/17$

(d) $2\sin^{-1}(3/5) - \tan^{-1}(17/31) = \pi/4$.

6. Prove that: $2\tan^{-1}(a) = \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \sin^{-1}\left(\frac{2a}{1+a^2}\right)$.

7. If $\sin^{-1}x + \sin^{-1}y = \pi/2$, then prove that: $\sin^{-1}x = \cos^{-1}y$.

8. If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi/2$, then prove that: xy + yz + zx = 1.

9. If $\tan^{-1}(x) + \tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}(3x)$, then prove that: x = 0, x = ± 1/2.

10. (a) If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi$, then prove that: x + y + z = xyz.

(b) If $\cot^{-1}(1/a) + \cot^{-1}(1/b) + \cot^{-1}(1/c) = \pi/2$, then prove that: ab + bc + ca = 1.

11. Prove that: $\sin^{-1}(1/\sqrt{5}) + \sin^{-1}(2/\sqrt{5}) = \pi/2$.

12. Prove that: $\tan(2\tan^{-1}x) = 2\tan(\tan^{-1}x + \tan^{-1}x^3)$.

13. Which is greater tan1 or $\tan^{-1}1$. Ans: tan1

14. Find the value of

$\sin(2\tan^{-1}(2/3)) + \cos(\tan^{-1}(\sqrt{3}))$ Ans: 37/26

15. Solve the following:

(a) $4\sin^{-1}x + \cos^{-1}x = \pi$ Ans: 1/2

(b) $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$

(c) $\tan(\cos^{-1}x) = \sin(\cot^{-1}\left(\frac{1}{25}\right))$

(d) $\sin^{-1}x + \tan^{-1}x = \pi/2$

16. Solve the following:

(a) $\tan^{-1}x + 2\cot^{-1}x = 2\pi/3$ Ans: $\sqrt{3}$.

(b) $\tan^{-1}\left(\frac{a+x}{a}\right) + \tan^{-1}\left(\frac{a-x}{a}\right) = \frac{\pi}{6}$. Ans: $\pm a\sqrt{2\sqrt{3}}$.

(c) $\sin^{-1}x + \sin^{-1}2x = \pi/3$ Ans: $\pm\frac{\sqrt{3}}{2\sqrt{7}}$.

(d) $\tan^{-1}2x + \tan^{-1}3x = n\pi + 3\pi/4$. Ans: 1, – 1/6.

17. Solve for x and y:

(a) $\sin^{-1}x + \sin^{-1}y = 2\pi/3$,
$\cos^{-1}x - \cos^{-1}y = \pi/3$ Ans: x = 1/2, y = 1.

(b) $\sin^{-1}x + \sin^{-1}y = \pi/2$,
$\cos^{-1}x - \cos^{-1}y = 0$ Ans: $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$.

18. Prove the following:

(a) $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0$

(b) $\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) = \tan^{-1}a - \tan^{-1}c$

(c) $\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) + \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = 2\tan^{-1}\left(\frac{a+b}{1+ab}\right)$

(d) $\tan^{-1}a + \tan^{-1}b = \cos^{-1}\left(\frac{1-ab}{\sqrt{(1+a^2)(1+b^2)}}\right)$

19. Prove that: $2\tan^{-1}\left(\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}\right) = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$.

20. (a) If $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \pi/6$, prove that: $x^2 = 2\sqrt{3}$.

(b) If $\sin^{-1}(x/5) + \operatorname{cosec}^{-1}(5/4) = \pi/2$, then prove that: x = 3.

(c) If $\tan^{-1}x + \tan^{-1}y = \pi/4$, then prove that: $x + y + xy = 1$.

21. If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, then find the value of x. Ans: $x = \pm 1/\sqrt{2}$.

22. Find the value of $\sin(\pi/3 - \sin^{-1}(-1/2))$. Ans: 1

23. Prove that:

(a) $\tan^{-1}(1/2) + \tan^{-1}(1/5) + \tan^{-1}(1/8) = \pi/4$.

(b) $\tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/3) + \tan^{-1}(1/8) = \pi/4$

(c) $\tan^{-1}(2/5) + \tan^{-1}(1/3) + \tan^{-1}(1/12) = \pi/4$

24. Show that:

(a) $\sin^{-1}(12/13) + \cos^{-1}(4/5) + \tan^{-1}(63/16) = \pi$

(b) $\cot^{-1}(7) + \cot^{-1}(8) + \cot^{-1}(18) = \cot^{-1}(3)$

25. Prove that: $\tan^{-1}\{(1 + x)/(1 - x)\} = \pi/4 + \tan^{-1}(x)$

26. If $\cos^{-1}x - \sin^{-1}x = 0$, then find x. Ans: $1/\sqrt{2}$.

27. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then find the value of $xy + yz + zx$. Ans: 3.

28. If two angles of a triangle are $\tan^{-1}2$ and $\tan^{-1}3$, prove that the third angle is $\pi/4$.

29. If $xy < 1$, then find the value of $\tan^{-1}x + \tan^{-1}y$.

30. Prove that: $\sin^{-1}(\cos\sin^{-1}x) + \cos^{-1}(\sin\cos^{-1}x) = \pi/2$.

31. Prove that: $\frac{1}{2}\tan^{-1}x = \cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

32. Prove that: $4(2\tan^{-1}(1/3) + \tan^{-1}(1/7)) = \pi$.

33. Solve for x:

(a) $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = 2\tan^{-1}x$. Ans: $\frac{a+b}{1-ab}$

(b) $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x$ Ans: $\frac{a+b}{1-ab}$

(c) $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{1+4x} = \tan^{-1}\frac{1}{x^2}$ Ans: 0, $\frac{1\pm\sqrt{5}}{2}$.

(d) $\sin^{-1}x + \sin^{-1}3x = \pi/3$ Ans: $\frac{\sqrt{3}}{2\sqrt{13}}$

(e) $\sin^{-1}6x + \sin^{-1}6\sqrt{3}x = -\pi/2$. Ans: – 1/12.

(f) $\tan^{-1}(x + 2) + \tan^{-1}(2 - x) = \tan^{-1}(2/3)$ Ans: –3, 3

(f) $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$. Ans: 0, 1/2

34. Evaluate: (a) $\tan\frac{1}{2}\left(\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$.

(b) $\tan\{2\tan^{-1}1/5 - \pi/4\}$.

(c) $\cos^{-1}x + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\}$.

Ans: (a) $\frac{1}{2}(3 - \sqrt{5})$ (b) –7/17 (c) $\pi/3$

35. If $\cos^{-1}(x/a) + \cos^{-1}(y/b) = A$, prove that: $x^2/a^2 - 2xy\cos A/ab + y^2/b^2 = \sin^2A$.

36. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that: $x^2 - y^2 - z^2 + 2yz\sqrt{1-x^2} = 0$.

37. Prove that:
$2\tan^{-1}\frac{1}{5} + \cos^{-1}\frac{7}{5\sqrt{2}} + 2\tan^{-1}\frac{1}{8} = \pi/4$.

38. Show that:

(a) $\tan^{-1}\left\{\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right\} = \frac{1}{2}\cos^{-1}\left(\frac{1+2\cos x}{2+\cos x}\right)$.

(b) $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \pi/2 + x/2$.

39. Find the value of $\sin(2\sin^{-1}(0.6))$. Ans: 0.96.

40. Find the domain of

(a) $\cos^{-1}(x^2 - 4)$ (b) $\sin^{-1}(-x^2)$

Ans:(a) $x \in [-\sqrt{5}, -\sqrt{3}]\cup[\sqrt{3}, \sqrt{5}]$ (b) $x \in [-1, 1]$.

41. Find the least and greatest value of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$. Ans: least $= \frac{\pi^2}{8}$, gratest $= \frac{5\pi^2}{4}$.

42. Prove that: $2(\tan^{-1}1 + \tan^{-1}1/2 + \tan^{-1}1/3) = \pi$.

43. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \pi^2/8$, then find the value of x. Ans: x = 1.

44. Solve: $\cos(\tan^{-1}x) = \sin(\cot^{-1}(3/4))$ Ans: 3/4.

45. Solve: $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}(8/31)$.
Ans: 1/4

46. Solve:
$\tan^{-1}(x - 1) + \tan^{-1}(x) + \tan^{-1}(x + 1) = \tan^{-1}3x$
Ans: ±1/2,0

47. Solve: $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = 2\pi/3$. Ans: $2 - \sqrt{3}$.

48. Solve:

$\tan^{-1}\sqrt{(x^2 + x)} + \sin^{-1}\sqrt{(x^2 + x + 1)} = \pi/2$ Ans: 0, - 1.

49. Solve: $\sin[\cot^{-1}(x + 1)] = \cos[\tan^{-1}x]$ Ans: - 1/2.

50. Prove that:

$\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$

51. Prove that:

$\sin^{-1}(4/5) + \sin^{-1}(5/13) + \sin^{-1}(16/65) = \pi/2.$

52. Prove that: $\cos\left[\sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\left(\frac{3}{2}\right)\right] = \frac{6}{5\sqrt{13}}.$

PREVIOUS YEAR'S BOARD QUESTION

2.2 BASIC CONCEPTS

VSA (1 Marks)

1. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{1}\left(\frac{1}{2}\right)$. (Foreign 2014)
2. Write the principal value of $\tan^{-1}\left[sin\left(-\frac{\pi}{2}\right)\right]$. (AI 2014C)
3. Find the value of the following: $\cot\left(\frac{\pi}{2} - 2cot^{-1}\sqrt{3}\right)$ (AI 2014C)
4. Write the principle value of: $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$. (Delhi 2013)
5. Write the value of $\tan^{-1}\left[2sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. (AI 2013)
6. Write the principle value of $\left[\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)\right]$. (Delhi 2013 C)
7. Write the principle value of $[\tan^{-1}(-\sqrt{3}) + \tan^{-1}(1)]$. (AI 2013 C)
8. Write the principle value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$. (Delhi 2012)
9. Using principle values, write the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$. (AI 2012C)
10. Evaluate: $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$. (Delhi 2011C)
11. Write the principle value of $\sin^{-1}\left(-\frac{1}{2}\right)$. (Delhi 2011C)
12. Using principle values, write the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. (AI 2011C, Delhi 2010)
13. Find the principle value of $\sin^{-1}\left(\frac{-1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$. (Delhi 2010)
14. Find the principle value of $\sec^{-1}(-2)$. (AI 2010)
15. Using the principle values, evaluate the following: $\tan^{-1}1 + \sin^{-1}\left(\frac{-1}{2}\right)$. (Delhi 2009C)
16. Find the principle value of $\tan^{-1}(-1)$. (Delhi 2008C)
17. Find the principle value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. (AI 2008C)
18. Find the value of the following: $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$. (AI 2007)

LA1 (4 Marks)

19. Prove that $\cos^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{56}{33}$. (AI 2013C)
20. Prove that: $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$ (AI 2012, Delhi 2010C, 2009C)
21. Prove that: $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$ (AI 2012, Delhi 2010)
22. Prove that: $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \pi$. (Foreign 2008)

2.3 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

VSA (1 Marks)

23. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x. (Delhi 2014)
24. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$. (AI 2014, Delhi 2012C)
25. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. (Delhi 2013)
26. Write the principle value of: $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$. (AI 2013, 2018)
27. Evaluate $\sin^{-1}\left(sin\frac{3\pi}{5}\right)$. (AI 2013C, Delhi 2009)
28. Find the principle value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$. (AI 2012)
29. Find the value of $\tan^{-1}\left(tan\frac{3\pi}{4}\right)$. (Delhi 2011)
30. Write the principle value of $\cos^{-1}\left(cos\frac{7\pi}{6}\right)$. (Delhi 2011)
31. Using principle values, evaluate the following. $\cos^{-1}\left(cos\frac{2\pi}{3}\right) + \sin^{-1}\left(sin\frac{2\pi}{3}\right)$. (AI 2011, 2009C, 2008)
32. Find the principle value of $\sin^{-1}\left(sin\frac{4\pi}{5}\right)$. (AI 2010)
33. If $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}(x) = \frac{\pi}{2}$, then find x. (Delhi 2010C)
34. If $\sin^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then find x. (Delhi 2010C)
35. Using principle value, find the value of $\cos^{-1}\left(cos\frac{13\pi}{6}\right)$. (AI 2010C)
36. If $\tan^{-1}(\sqrt{3}) + \cot^{-1}(x) = \frac{\pi}{2}$, then find x. (AI 2010C)
37. Show that, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$. (Foreign 2008)
38. Write into the simplest form: $\tan^{-1}\left(\frac{\sqrt{1+sinx} + \sqrt{1-sinx}}{\sqrt{1+sinx} - \sqrt{1-sinx}}\right)$. (Delhi 2007)

LA1 (4 Marks)

39. Prove that: $\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$. (Delhi 2017)
40. If $\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of x. (AI 2017)

41. Solve for x: $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.
(Delhi 2016, 2014C, Foreign2015, AI 2009)

42. Prove that:
$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.
(Delhi 2016, 2008, 2008C, AI 2010, 2009C, 2008)

43. Solve the equation for x:
$\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$ (AI 2016)

44. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, prove that:
$\frac{x^2}{a^2} - 2\frac{xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$. (AI 2016)

45. Solve for x: $\tan^{-1}\left(\frac{x-2}{x-1}\right) + \tan^{-1}\left(\frac{x+2}{x+1}\right) = \frac{\pi}{4}$
(Foreign 2016)

46. Prove that: $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$.
(Foreign 2016, Delhi 2014, 2014C, 2011, AI2009)

47. If $\sin[\cot^{-1}(x + 1)] = \cos(\tan^{-1}x)$, then find x.
(Delhi 2015)

48. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then find x. (Delhi 2015)

49. Prove the following:
$\cot^{-1}\left(\frac{xy+1}{x-y}\right) + \cot^{-1}\left(\frac{yz+1}{y-z}\right) + \cot^{-1}\left(\frac{zx+1}{z-x}\right) = 0$,
$(0 < xy, yz, zx < 1)$ (AI 2015)

50. Solve for x: $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\frac{8}{31}$.
(AI 2015, Foreign 2008)

51. If $\tan^{-1}\left(\frac{1}{1+1.2}\right) + \tan^{-1}\left(\frac{1}{1+2.3}\right) + \ldots + \tan^{-1}\left(\frac{1}{1+n.(n+1)}\right)$
$= \tan^{-1}\theta$, then find the value of θ.
(Foreign 2015)

52. Solve for x: $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$.
(Delhi 2015C, 2013C, 2009, 2008, AI 2012C, 2009C).

53. Prove that: $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$. (Delhi 2015C)

54. Prove that:
$2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$. (AI 2015 C)

55. Solve for x: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$, $x > 0$.
(AI 2015C, 2014C, 2010C, Delhi 2008C, Foreign 2008)

56. Prove that: $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$
(Delhi 2014)

57. Prove that: $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$, $\frac{-1}{\sqrt{2}} \le x \le 1$
(AI 2014, 2011, 2010C)

58. If $\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, find the value of x.
(AI 2014)

59. Solve for x: $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$.
(Foreign 2014, AI 2013)

60. Prove that: $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$.
(Foreign 2014)

61. Prove that: $\cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\} = \frac{\pi}{3}$. (AI 2014C)

62. Solve for x: $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$.
(AI 2014C, Delhi 2009C)

63. Prove that: $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{36}{85}$.
(AI 2014C, Delhi 2012, 2010C)

64. Find the value of the following:
$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$, $|x| < 1$, $y > 0$ and
$xy < 1$. (Delhi 2013)

65. Prove that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.
(Delhi 2013, 2012C, 2008C, AI 2011)

66. Show that: $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$. (AI 2013)

67. Write the value of the following:
$\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{a-b}{a+b}\right)$ (Delhi 2013C)

68. Prove that: $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$. (Delhi 2013C)

69. Solve for x: $\sin^{-1}(1 - x) - 2\sin^{-1}x = \frac{\pi}{2}$. (AI 2013C)

70. Prove that: $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. (Delhi 2012)

71. Prove the following: $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$.
(AI 2012)

72. Solve for x: $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.
(Delhi 2012C, 2009C, 2008, AI 2010, 2008)

73. Prove that: $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$.
(Delhi 2012C, AI 2009C)

74. Find the value of: $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$. (Delhi 2011)

75. Prove that: $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$.
(AI 2011, Delhi 2009)

76. Prove that: $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$. (Delhi 2011C)

77. Solve for x: $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$, $-1 < x < 1$.
(Delhi 2011C)

78. Prove that: $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\tan^{-1}\frac{4}{3}$ (AI 2011C)

79. Solve for x: $\cos(2\sin^{-1}x) = \frac{1}{9}$, $x > 0$. (AI 2011C)

80. Prove that: $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in (0, 1)$. (Delhi 2010)

81. Prove that:
$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$. (Delhi 2010)

82. Prove that: $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$.
(AI 2010)

83. Prove that: $\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.
(AI 2010)

84. Solve for x: $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}$; $0< x < \sqrt{6}$. (Delhi 2010C)

85. Solve for x: $\tan^{-1}(x + 2) + \tan^{-1}(x – 2) = \tan^{-1}\left(\frac{8}{79}\right)$; $x > 0$.
(Delhi 2010C)

86. Prove that: $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$. (AI 2010C)

87. Solve for x: $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$. (AI 2010C)

88. Prove that: $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$.
(Delhi 2009C)

89. Prove that: $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{4}{7}$. (Foreign 2008)

90. Solve for x: $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1}x$, $0 < x < 1$.
(Delhi 2008C)

91. Write into the simplest form: $\cot^{-1}\left(\sqrt{1+x^2} - x\right)$.
(Delhi 2007)

92. Prove that: $\frac{9\pi}{8} – \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$. (AI 2007)

93. Prove that $3\sin^{-1}x = \sin^{-1}(3x – 4x^3)$, x lies in the interval [- 1/2, 1/2]. (AI 2018)

94. Evaluate: $\tan^{-1}(\sqrt{3}) – \cot^{-1}(– \sqrt{3})$. (CBSE 2018)

ANSWERS

1. π 2. – π/4 3. $\sqrt{3}$ 4. 11 π/12 5. π/3.
6. 5 π/6 7. – π/12. 8. 2 π/3 9. 2 π/3 10. 1
11. – π/6 12. – π/3 13. π/2 14. 2 π/3 15. π/12
16. – π/4 17. π/6. 18. 3 π/4 23. 1/5 24. 1
25. 5/12 26. – π/2 27. 2 π/5 28. – π/3 29. – π/4
30. 5 π/6 31. Π 32. π/5 33. 1/3 34. ½ 35. π/6
36. $\sqrt{3}$ 38. x/2 40. $\pm\sqrt{\frac{17}{2}}$ 41. π/4 43. 0, ½ 45. $\pm\sqrt{\frac{7}{2}}$
47. – ½ 48. 3 π/4, - π/4 50. ¼, - 8 51. n/(n + 2)
52. 1/6, - 1 55. $1/\sqrt{3}$ 58. $\pm\sqrt{\frac{7}{2}}$59. ¾, - ¾ 62. $\sqrt{3}$
64. (x + y)/(1 – xy) 67. π /4 69. 0 72. $\pm 1/\sqrt{2}$. 74. π/4
77. 2 - $\sqrt{3}$ 79. 2/3 84. 1 85. ¼, 87. 1, - 1 90. (0, 1)
91. π/2 – $\cot^{-1}x/2$ 94. - π/2.

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